The University of Georgia
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EMAT 6680 - Assignment 2

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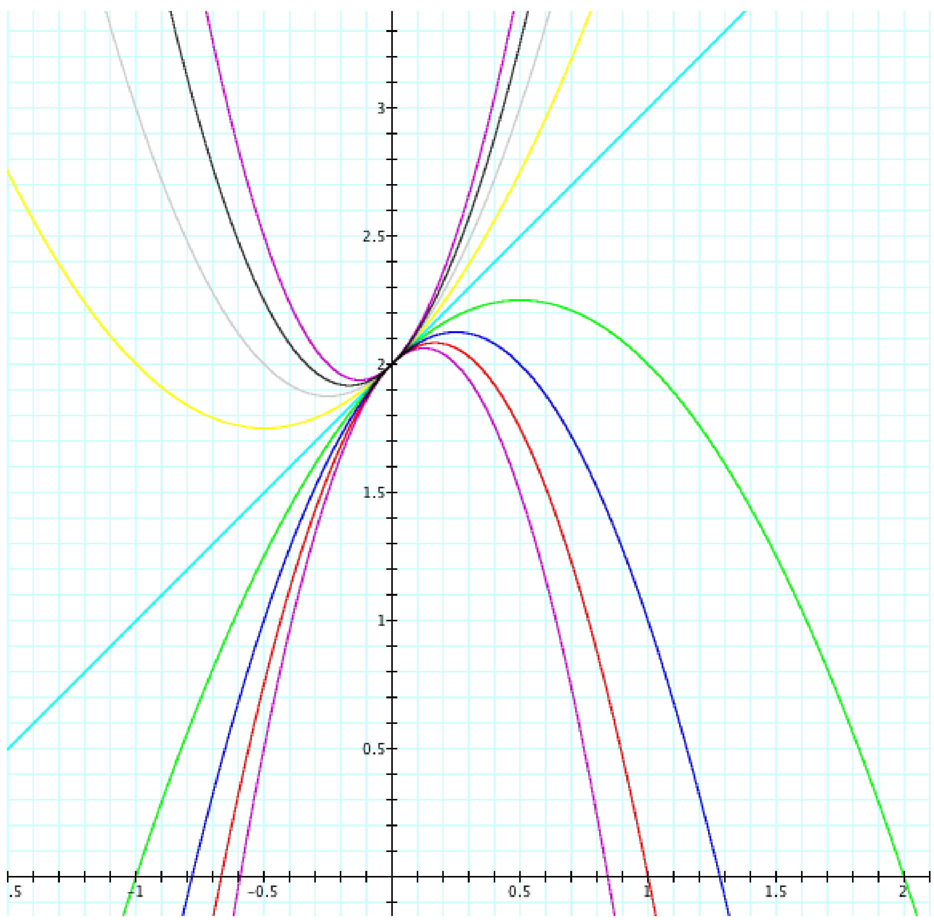
Question: Given the function $f(x) = ax^2 + x + 2$, explore the various behaviors as a changes. Then discuss what happens to other graphs as the x coefficient and the constant term change.

To begin, let's graph this function for the values of $a \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$:



First, we note that the functions with positive a bend upward, while those with negative a bend downward. Of course, this behavior is true for any quadratic polynomial. Obviously, the $a = 0$ case is special in that it becomes a line with slope 1.

Lemma 1. $\lim_{x\to\pm\infty} ax^2 + bx + c = \infty$ *for* $a > 0$ *and* $\lim_{x\to\pm\infty} ax^2 + bx + c = -\infty$ *for* $a < 0$.

Proof. We will give a proof for the first case and simply note that the other case is similar.

$$\begin{aligned}
\lim_{x\to\pm\infty} ax^2 + bx + c &= \lim_{x\to\pm\infty} ax^2\left(1 + \frac{b}{ax} + \frac{c}{ax^2}\right) \\
&= \lim_{x\to\pm\infty} (ax^2) \lim_{x\to\pm\infty} \left(1 + \frac{b}{ax} + \frac{c}{ax^2}\right) \\
&= \infty \times 1 \\
&= \infty
\end{aligned}$$

□

Next, if we look at the graph above, then we notice that the various graphs for different values of a all share a unique common point. This phenomenon is actually true in general, so let's explore this further.

Lemma 2. *The point* $(0, 2)$ *is the only point common to all functions of the form* $f(x) = ax^2 + x + 2$.

Proof. The fact that $(0, 2)$ is a common point is clear, so we just need to show uniqueness. Suppose (x, y) is a point which is common to all functions of the form $f(x) = ax^2 + x + 2$. Then for any two distinct such functions, we have $a_1x^2 + x + 2 = a_2x^2 + x + 2$. Therefore, we see that $a_1x^2 = a_2x^2$, but since a_1 and a_2 are distinct, it must be the case that $x = 0$. □

Next, let's explore the vertex of the parabolas ($a \neq 0$). By a little calculus, we know that the vertex is either the absolute maximum or minimum of the function, hence its $x-$coordinate must be the solution to the equation $2ax + 1 = 0$. In the case of $f(x) = ax^2 + x + 2$, it must be of the form

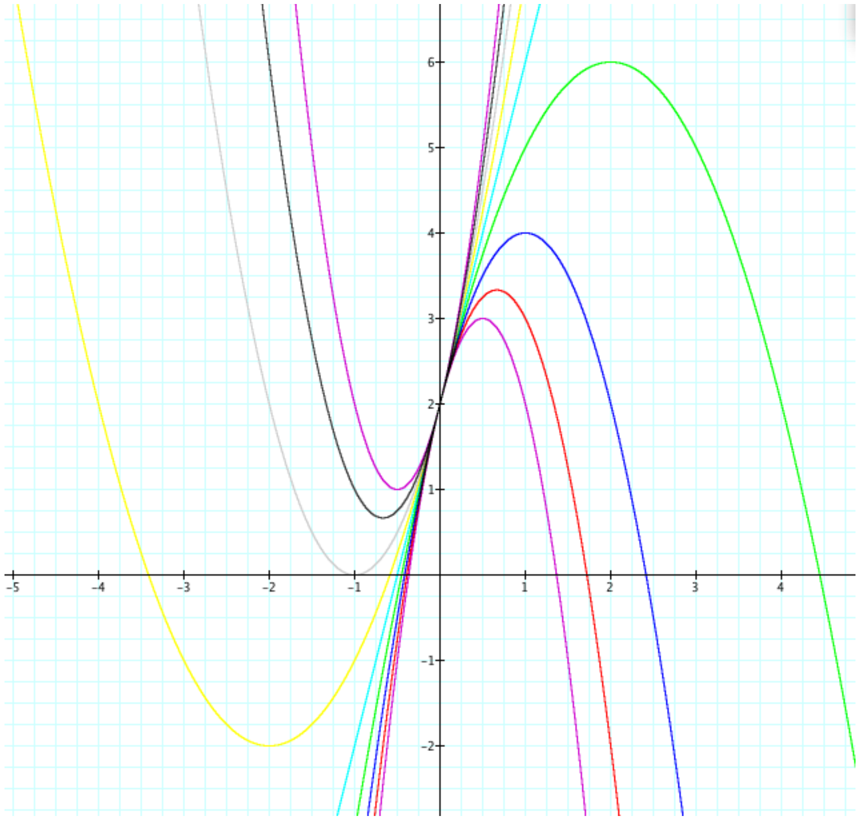
$$\left(\frac{-1}{2a}, \frac{1}{4a} + \frac{-1}{2a} + 2\right) = \left(\frac{-1}{2a}, \frac{8a-1}{4a}\right).$$

Now, what's interesting is that we have another way of recovering the point in common to all functions of the form $f(x) = ax^2 + x + 2$:

$$\lim_{a\to\pm\infty} \left(\frac{-1}{2a}, \frac{8a-1}{4a}\right) = (0, 2).$$

Therefore, the common point is the limit of the vertices.

Now, let's explore what happens to the graph when we change the x coefficient from 1 to 4.



First off, we see that the parabolas become steeper in the sense that the tangent line at the point $(0, 2)$ has gone from a line with slope 1 to a line with slope 4. Then we ask ourselves if this plot gives us what we should have expected. Actually, it should make sense in the following way: The line obtained from $f(x)$ by setting $a = 0$ describes the behavior of the other functions since it is tangent to all of them at the common point $(0, 2)$. This tells us that if we want to graph the various $f(x) = ax^2 + 4x + 2$, then we should first graph $4x + 2$ and we will have a clear picture of what the functions will look like. Finally, we conclude by observing that if the constant term is changed then functions shift vertically up or down depending on the sign of the constant term, which is true of any function $f(x)$.